

Vectors and geometric proof

Topic cliff notes / objectives, key ideas and examples

Learning objectives

- Add, subtract and reverse vectors along a path.
- Prove parallelism using a non-zero scalar multiple.

Key ideas

- Subtract position vectors to find a displacement.

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b} \quad \Rightarrow \quad \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

- Following a path adds vectors; reversing direction changes the sign.

$$\overrightarrow{BA} = -\overrightarrow{AB}$$

- A midpoint averages the two position vectors.

$$\overrightarrow{OM} = \frac{\mathbf{a} + \mathbf{b}}{2}$$

- For non-zero vectors, a non-zero scalar multiple proves parallelism. A positive scalar preserves direction.

Worked example

O and A are distinct. M is the midpoint of AB; N is the midpoint of OB. Prove NM is parallel to OA and half its length.

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}$$

$$\overrightarrow{OM} = \frac{\mathbf{a} + \mathbf{b}}{2}, \quad \overrightarrow{ON} = \frac{\mathbf{b}}{2}$$

$$\overrightarrow{NM} = \overrightarrow{OM} - \overrightarrow{ON}$$

Answer

$$\overrightarrow{NM} = \frac{\mathbf{a}}{2} = \frac{1}{2}\overrightarrow{OA}$$

Quick example

M is the midpoint of AB. Find the vector from A to M.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

Answer

$$\overrightarrow{AM} = \frac{\mathbf{b} - \mathbf{a}}{2}$$

Watch out

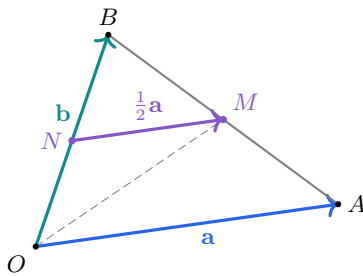
Keep the order of subtraction correct. Position vectors start at the origin; a displacement vector may start elsewhere.



Vectors: see the proof

Visual guide / paths, midpoints and parallel directions

A midpoint proof



M is the midpoint of AB and N is the midpoint of OB . The purple arrow has the same direction as OA and half its length.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, with $O \neq A$.

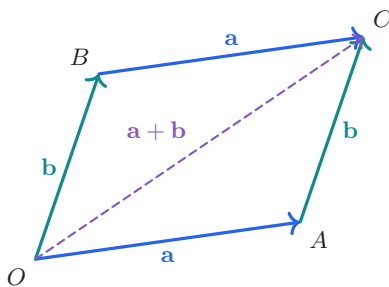
$$\overrightarrow{OM} = \frac{\mathbf{a} + \mathbf{b}}{2}, \quad \overrightarrow{ON} = \frac{\mathbf{b}}{2}.$$

Subtract position vectors *end minus start*:

$$\overrightarrow{NM} = \overrightarrow{OM} - \overrightarrow{ON} = \left[\frac{1}{2} \mathbf{a} \right].$$

So $NM \parallel OA$ and $NM = \frac{1}{2}OA$. The positive scalar $\frac{1}{2}$ keeps the arrow's direction.

A parallelogram: two paths give the same result



The colour of an arrow marks its vector: translated arrows keep both direction and length.

Following $O \rightarrow A \rightarrow C$ adds the vectors:

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \boxed{\mathbf{a} + \mathbf{b}}.$$

The other route $O \rightarrow B \rightarrow C$ gives $\mathbf{b} + \mathbf{a}$, the same displacement.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{BA} = \mathbf{a} - \mathbf{b}.$$

Reversing an arrow changes its sign.

Midpoint displacement

For M halfway from A to B , $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$. A position vector starts at O ; AM starts at A .

Watch out: a scalar multiple proves parallel directions only for non-zero vectors. Label the arrow's start and end before subtracting.

